Perfect Polyominoes

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Abstract

The concept of perfect numbers may be extended to polyominoes. In the search for perfect polyominoes, numerous charming and motivating examples can be found, while some sizes may be ruled out on number theoretic grounds. We demonstrate the infinitude of perfectominoes, share perfectominoes of odd size, and introduce open questions.

Project Description

When congruent copies of a polyomino can be combined to form another polyomino, the former is said to be a **factor** of the latter. This defines a divisor relation on the set of polyominoes, which is isomorphic to the usual notion of divisibility of natural numbers for the subset of $1 \times n$ rectangles, or **stick polyominoes**. Looking at the complete set of polyominoes, we can generalize certain aspects of number theory. For example, a polyomino might be prime, or two polyominoes might have a least common multiple. In this vein, a **perfect polyomino** (or perfectomino) is one that can be reassembled as a sum of its proper factors.

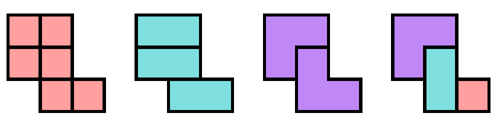

Figure 1: The Q-hexomino is perfect.

For any perfect number n (of which 48 are currently known), the stick polyomino of size n is perfect. However, polyominoes of perfect size are not necessarily perfect. Of the 35 free hexominoes seen in Figure 2, only 10 are perfect. While a hexomino can only have proper factors of size 1, 2, or 3, it may have multiple tromino factors, or its shape may preclude domino or tromino factors. In fact, there are five hexominoes that are prime. Further, there is at least one prime polyomino for any size.

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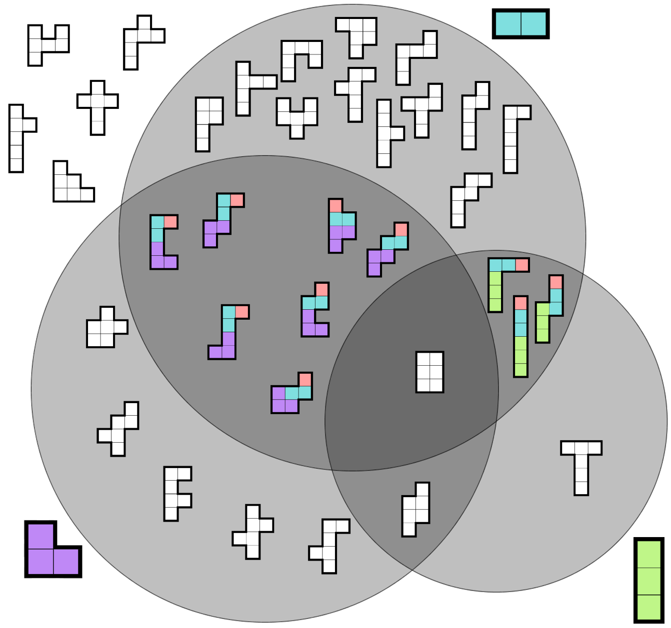

Figure 2: The 35 free hexominoes, sorted by their factors.

By repeating or dropping factors as described above, it is possible for a polyomino to be perfect even when its size is imperfect. Although 18 is an abundant number, for example, the 18-omino in Figure 3 is divisible by both trominoes and no hexomino. This corrects the factor sum and yields a perfectomino.

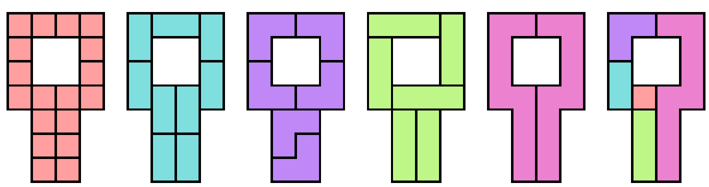

Figure 3: "Lanier's Looking Glass," a perfect 18-omino.

Some sizes of polyominoes may be ruled out as candidates for perfectominoes due to number theoretic considerations.

- No perfectomino exists for any size a power of a prime p. Looking at the sizes of potential factors mod p, we get a sum of 1 mod p – since the monomino divides every polyomino – while the polyomino's size is 0 mod p.
- Take any prime p, and let K be the number of distinct free p-ominoes. There are no perfectominoes of size pq, where q is a prime greater than $Kp + 1$. (proof omitted)

There are some sizes for which a numerical partition makes a perfectomino conceivable, but no such polyomino exists. For example, an exhaustive computer search has shown that no perfect 21-omino exists, though 1+3+3+7+7 is a valid partition of 21 as a sum of its proper factors. Though the study of perfect numbers is ancient, to date only 48 have been found, none of them odd. Perfect polyominoes of odd-size do exist, however. The smallest constructed example has size 45.

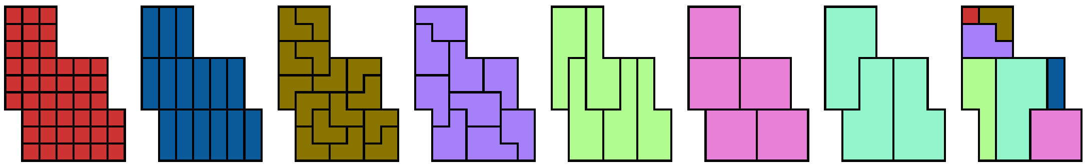

Figure 4: The smallest known perfect polyomino of odd size.

Euclid provided a method for constructing perfect numbers, but it relies on the discovery of Mersenne primes. We offer a similar constructive method for generating perfectominoes. For any even perfect number k, we begin with the perfect stick polyomino of length k. By gluing 2^n copies of this perfect stick together in a zig-zag with regular bends, we construct a perfect $(2^n \cdot k)$-omino. Hence, we demonstrate there are infinitely many perfect polyominoes.

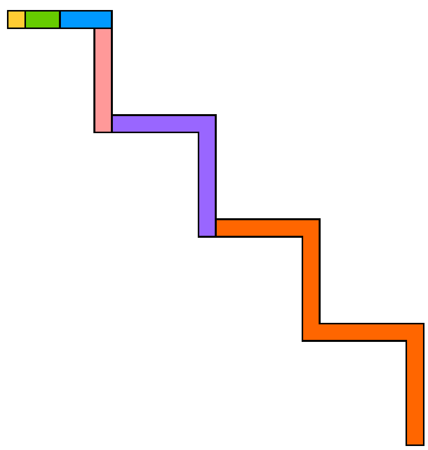

Figure 5: A perfect zig-zag $(2^3 \cdot 6)$-omino.