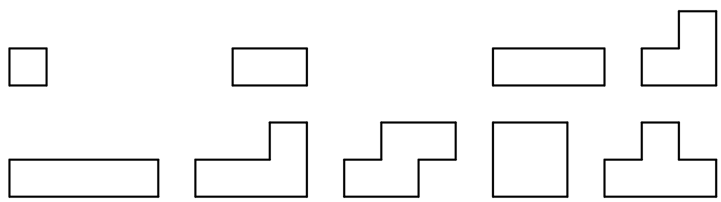
Polyomino Number Theory (I)

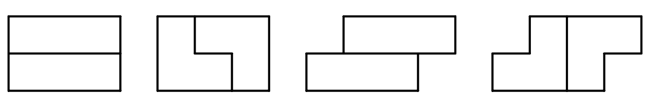
Andris Cibulis, Andy Liu and Bob Wainwright

Polyominoes are connected plane figures formed of joining unit squares edge to edge. We have a monomino, a domino, two trominoes named $\boldsymbol{I}$ and $\boldsymbol{V}$, and five tetrominoes named $\boldsymbol{I}$, $\boldsymbol{L}$, $\boldsymbol{N}$, $\boldsymbol{O}$ and $\boldsymbol{T}$, respectively.

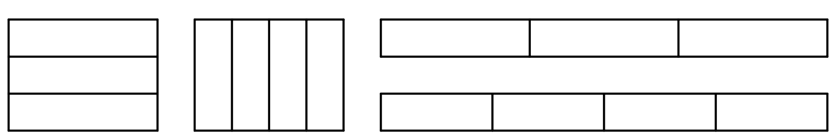


A polyomino $\boldsymbol{A}$ is said to **divide** another polyomino $\boldsymbol{B}$ if a copy of $\boldsymbol{B}$ may be assembled from copies of $\boldsymbol{A}$. We also say that $\boldsymbol{A}$ is a **divisor** of $\boldsymbol{B}$, $\boldsymbol{B}$ is **divisible** by $\boldsymbol{A}$, and $\boldsymbol{B}$ is a **multiple** of $\boldsymbol{A}$. The monomino divides every polyomino.

A polyomino is said to be a **common divisor** of two other polyominoes if it is a divisor of both. It is said to be a **greatest common divisor** if no other common divisor has greater area. Note that we say a greatest common divisor rather than the greatest common divisor since it is not necessarily unique. For instance, the two hexominoes below have both the $\boldsymbol{I}$–tromino and the $\boldsymbol{V}$–tromino as their greatest common divisors.



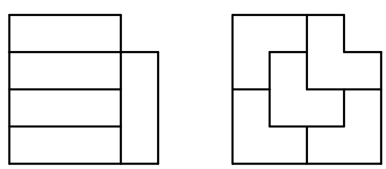
When two polyominoes have at least two greatest common divisors, each greatest common divisor is clearly not divisible by any of the others. However, even if a unique greatest common divisor exists, it is still not necessarily divisible by the other common divisors. For instance, the two dodecominoes below have the $\boldsymbol{I}$–tetromino as their unique greatest common divisor, but it is not divisible by the $\boldsymbol{I}$–tromino which is also a common divisor.



Any two polyominoes have a greatest common divisor, since we can always fall back on the monomino. When the greatest common divisor is the monomino, we say that these two polyominoes are **relatively prime** to each other. The monomino is relatively prime to every other polyomino. A **prime** polyomino is one which is divisible only by itself and the monomino, and it is also relatively prime to every other polyomino. Note that the monomino is not considered to be a prime polyomino.

If the area of a polyomino is a prime number, then it must be a prime itself. The converse is not true. The smallest counter-example is the $\boldsymbol{T}$–tetromino. It has area 4, but is a prime polyomino.

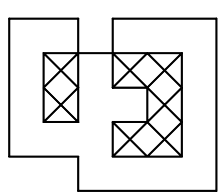
A polyomino is said to be a **common multiple** of two other polyominoes if it is a multiple of both. If two polyominoes have common multiples, they are said to be **compatible**. A **least common multiple** of two compatible polyominoes is a common multiple with minimum area. As shown earlier, the $\boldsymbol{I}$–tromino and the $\boldsymbol{V}$–tromino have at least two least common multiples. Clearly, neither multiple divides the other. These two trominoes even have a common multiple whose area is not divisible by 6, the area of their least common multiple.



However, the area of every common multiple of the $\boldsymbol{I}$–tromino and the $\boldsymbol{I}$–tetromino must be a multiple of 12, the area of their least common multiple.

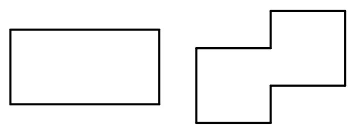
Given two small polyominoes, it is a trivial matter to determine all common divisors of them. It is a different situation with common multiples. To determine whether they are even compatible is often an interesting question. Finding the area of a least common multiple of two compatible polyominoes can also be challenging.

The monomino is trivially compatible with every polyomino. This property is not shared even by the domino, which is incompatible with the icosomino below. Thus compatibility is not a transitive relation.

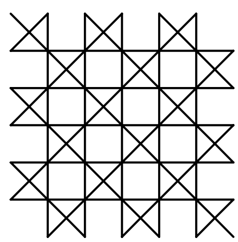


Suppose we wish to find a least common multiple of the O–tetromino with either the T–tetromino or the N–tetromino. Clearly, the area of any multiple of a tetromino is a multiple of 4. Since the tetrominoes in question are distinct, the smallest possible area of a common multiple is 8.

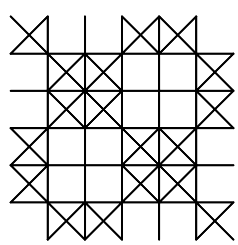
Note that two copies of the O–tetromino can abut in essentially two ways as shown below. Neither figure can be assembled from copies of either the T–tetromino or the N–tetromino. Hence a common multiple has area at least 12.



If we paint the squares of the infinite grid black and white in the usual checkerboard fashion as shown below, then three copies of the O–tetromino always cover an even number of white squares, while three copies of the T–tetromino always cover an odd number of white squares. Hence they have no common multiples with area 12.

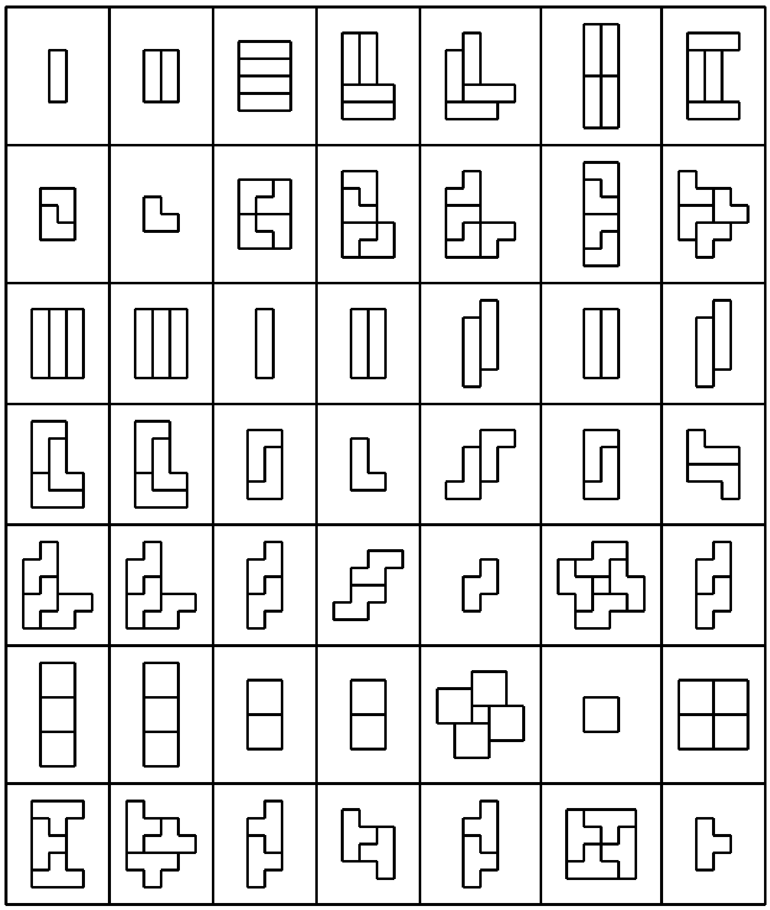


If we paint the squares of the infinite grid black and white in the checkered pattern shown below, then three copies of the O–tetromino always cover an even number of white squares, while three copies of the N–tetromino always cover an odd number of white squares. Hence they have no common multiples with area 12 either.



It follows that in both cases, the minimum area of a common multiple is 16. It turns out that such least common multiples exist.

The chart below gives a least common multiple of each pair of trominoes and tetrominoes. The polyominoes are featured along the main diagonal. The figure in the i^{th} row and the j^{th} column shows how a least common multiple of the i^{th} and j^{th} polyominoes can be constructed from the i^{th} polyomino.



Note that the minimum possible area is attained in all but two cases, between the O–tetromino on the one hand, and the T–tetromino and the N–tetromino on the other. We have dealt with these cases earlier.

The structure considered in this paper is an example of a *Normed Division Domain* considered by Solomon W. Golomb in his paper with that title, published on pages 680 to 686 in Volume 88 of the American Mathematical Monthly in 1981.

Andris Cibulis
Riga
Latvia
Andrejs.Cibulis@mii.lu.lv

Andy Liu
Edmonton
Canada
aliu@math.ualberta.ca

Bob Wainwright
New Rochelle
USA
life1ine@aol.com

Note that the fifth character in "line1ine" is the digit 1 and not the letter 'l'